An exactly solvable quantum-lattice model with a tunable degree of nonlocality

Miloslav Znojil

Nuclear Physics Institute ASCR,

250 68 Řež, Czech Republic

e-mail: znojil@ujf.cas.cz

Abstract

A non-Hermitian $N-$site-lattice Hamiltonian H with Laguerre-polynomial right eigenvectors and real energies is made self-adjoint in an *ad hoc* Hilbert space $\mathcal{H}^{(S)}$. The necessary physical inner products are defined via alternative $k-$parametric $(2k+1)-$diagonal metrics $\Theta = \Theta_k$ constructed, in closed form, at $k = 0, k = 1, k = 2$ and $k = 3$. The value of k is interpreted as a degree of non-locality of the model.

1 Introduction

1.1 The Laguerre-polynomial quantum model

It is well known that the Laguerre polynomials[1] $L(n,a,z)$ may *formally* be arranged in an infinite-dimensional Dirac-ket-like column vector

$$|\psi^{(\infty)}\rangle = \left[\begin{array}{c} L(0,a,z)\ (=1) \\ L(1,a,z)\ (=a+1-z) \\ L(2,a,z)\ \left[=\frac{1}{2}(a+1)(a+2)-(a+2)\,z+\frac{1}{2}\,z^2\right] \\ \vdots \end{array} \right] \tag{1}$$

which satisfies, line-by-line and at any complex $z \in \mathbb{C}$, the infinite linear algebraic system of equations

$$H^{(\infty)}(a)\,|\psi^{(\infty)}\rangle = z\,|\psi^{(\infty)}\rangle \tag{2}$$

which resembles the Schrödinger equation of quantum theory and where

$$H^{(\infty)}(a) = \left[\begin{array}{ccccc} a+1 & -1 & 0 & 0 & \ldots \\ -a-1 & a+3 & -2 & 0 & \ldots \\ 0 & -a-2 & a+5 & -3 & \ddots \\ 0 & 0 & -a-3 & a+7 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{array} \right] . \tag{3}$$

One can immediately formulate at least three reasons against $H^{(\infty)}(a)$ of Eq. (3) being interpreted as a quantum Hamiltonian:

- (pragmatic reason) the set of all complex $z \in \mathbb{C}$ does not look like a good candidate for a phenomenologically meaningful spectrum;

[1] our present denotation $L(n,a,x)$ and normalization $L(-1,a,x)=0,\ L(0,a,x)=1$ are taken from the symbolic-manipulation langauge MAPLE [1]; up to a factor the same polynomials are denoted as $L_n^a(x)$ in Ref. [2] or as $L_n^{(a)}(x)$ in Ref. [3]

- (mathematical reason) in the current, "friendly" Hilbert space ℓ^2 (to be denoted by symbol $\mathcal{H}^{(F)}$, cf. Appendix) the norm of the wave-function candidate (1) would be infinite;

- (physical reason) the candidate $H^{(\infty)}(a)$ for the Hamiltonian of the system is manifestly non-Hermitian.

1.2 The finite-dimensional Laguerre-polynomial quantum model

Let us replace the matrix-resembling array $H^{(\infty)}(a)$ by its finite-dimensional, truncated version

$$H^{(N)}(a) = \left[\begin{array}{ccccc} a+1 & -1 & 0 & \ldots & 0 \\ -a-1 & a+3 & -2 & \ddots & \vdots \\ 0 & -a-2 & a+5 & -3 & 0 \\ \vdots & \ddots & \ddots & \ddots & -N+1 \\ 0 & \ldots & 0 & -a-N+1 & a+2N-1 \end{array} \right] . \quad (4)$$

A return of the finite-dimensional descendant of Eq. (2)

$$H^{(N)}(a)\,|\psi_n^{(N)}\rangle = E_n^{(N)}(a)\,|\psi_n^{(N)}\rangle\,, \quad n = 1, 2, \ldots, N-1 \quad (5)$$

to the status of Schrödinger equation finds a new support in a few suddenly emerging encouragements:

- (pragmatic encouragement) after truncation, the set of the admissible eigenvalues shrinks from the whole complex plane of z to the discrete and non-degenerate $N-$plet of strictly real zeros of the elementary secular equation

$$L(N, a, z) = 0\,, \quad z = E_n\,, \quad n = 0, 1, \ldots, N-1\,; \quad (6)$$

- (mathematical encouragement) after truncation (and in both the Hilbert spaces $\mathcal{H}^{(F,S)}$ defined in Appendix), the finite array

$$|\psi_n^{(N)}(a)\rangle = \begin{bmatrix} L(0,a,E_n) \\ L(1,a,E_n) \\ \vdots \\ L(N-1,a,E_n) \end{bmatrix} \tag{7}$$

  may very easily be normalized, say, to one.

An obstacle which seems to survive is the third, non-Hermiticity point and counterargument. In what follows we intend to show that this argument is misleading and that in a way summarized in Appendix the Hermiticity of Hamiltonian $H^{(N)}(a)$ may be achieved and guaranteed in a large number of alternative physical Hilbert spaces $\mathcal{H}^{(S)}$ where the inner products are defined in terms of the so called metric operator Θ. We shall also show that for our particular model these spaces (i.e., the metric-operator matrices) may be constructed by purely non-numerical means.

Table 1: A sample of the dimension- and parameter-dependence of the energy spectra $\{E_n(a)\}$ of Hamiltonian (4).

parameters		energies				
N	a	$E_0(a)$	$E_1(a)$	...	$E_{N-2}(a)$	$E_{N-1}(a)$
6	1.0	0.5276681217	1.796299810	...	11.23461043	17.64596355
	2.0	0.8899410156	2.433144232	...	12.60041387	19.26204255
	3.0	1.296419203	3.093998381	...	13.94134537	20.83985455
9	1.0	0.3681784529	1.243357962	...	20.38218199	28.11834338
	2.0	0.6318537723	1.712163195	...	21.90120660	29.82533613
	3.0	0.9343511232	2.208578822	...	23.39499254	31.50012806

1.3 A phenomenological addendum

Our present solvable toy-model (4) is just a very special case of the broad class of the $N-$dimensional tridiagonal-matrix Hamiltonians of the form

$$\hat{H}^{(N)} = \begin{bmatrix} a_1 & c_1 & 0 & 0 & \ldots & 0 \\ b_2 & a_2 & c_2 & 0 & \ldots & 0 \\ 0 & b_3 & a_3 & c_3 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & b_{N-1} & a_{N-1} & c_{N-1} \\ 0 & \ldots & 0 & 0 & b_N & a_N \end{bmatrix} \tag{8}$$

which describe the one-dimensional single-particle $N-$site quantum-lattice dynamics reduced to the mere nearest-neighbor interaction. The kinematical aspect of virtually all of these models $\hat{H}^{(N)}$ is, typically, characterized by the one-to-one correspondence between the matrix index $s = 1, 2, \ldots, N$ and a discrete coordinate q_s of the "site". In a more dimensional case these sites may be numbered by an *ad hoc* multiindex, $s \to \vec{s}$, but just the simplest one-dimensional case is to be considered in what follows.

At the very start of our present application of the theoretical innovation of the formalism using $\Theta \neq I$ in combination with Hamiltonian (8) and/or its special case (4) let us emphasize that such an effort may even be supported by a purely pragmatic motivation. The numerical sample of zeros $z = E_n^{(N)}(a)$ of Eq. (6) as displayed in Table 1 persuades us, for example, that each of the $N-$plets of the present toy-model bound-state energies really looks very much like a realistic phenomenological spectrum.

Besides the similar intuitive argument it is equally important to keep in mind that the necessary modifications and adaptations or deformations of the spectrum are controlled not only by the freely variable coupling *alias* self-coupling *alias* asymmetry-strength $a > 0$ but also by the very choice of the *ad hoc* model-space dimension $N = 1, 2, \ldots$.

This means that even on the level of practical applicability, our unusual

model (4) might find its place among the other standard fitting tools for the description of experimental quantum spectra, especially when they are not explained by the conventional interactions yielding vibrations (like harmonic oscillator) or rotations (mainly in more spatial dimensions). Thus, although the core of our present message should be predominantly methodical and mathematical, even its purely pragmatic aspects might prove relevant and/or useful.

2 The concept of position in quantum lattice

In the vast majority of phenomenological quantum-lattice models the Hamiltonian is assumed Hermitian in the most common $N-$dimensional vector space, i.e., we have $\hat{H}^{(N)} = \left(\hat{H}^{(N)}\right)^{\dagger}$. In parallel, the coordinates q_s are assumed real and observable so that, in the Schrödinger's "mode of description" [4], the corresponding quantum operator of position is most often represented by a diagonal and time-independent $N-$dimensional matrix

$$\hat{\mathfrak{q}} = \begin{pmatrix} q_1 & & & \\ & q_2 & & \\ & & \ddots & \\ & & & q_N \end{pmatrix} . \tag{9}$$

Using the standard Dirac's notation the time-dependent Schrödinger equation then reads

$$\mathrm{i}\partial_t \, |\psi(t)\rangle = \hat{H}^{(N)} \, |\psi(t)\rangle \tag{10}$$

and controls the evolution of the system prepared in a normalized pure state at time $t_{prep} = 0$. At the time of measurement $t > 0$ one evaluates the wave function $\langle q_s|\psi(t)\rangle = \psi(t,s)$ and determines the probability

$$\varrho(t,s) = |\langle q_s|\psi(t)\rangle|^2 = \psi^*(t,s)\,\psi(t,s) \tag{11}$$

of finding the particle (or quasi-particle) at the $s-$th site q_s.

2.1 The loss of the measurability of position in the so called $\mathcal{PT}-$symmetric Quantum Mechanics

In the recent literature there emerged an interesting modification of the whole paradigm (cf. its compact summary in Appendix). It has been developed, mainly, within the framework of the so called $\mathcal{PT}-$symmetric quantum mechanics (cf., e.g., comprehensive reviews [5, 6] and also [7] or [8]). In what follows we intend to describe one of applications of this new paradigm to our exactly solvable single-particle $N-$site quantum-lattice model (4) where the spectrum remains real but where the Hamiltonian matrix itself appears *manifestly non-Hermitian* in $\ell^2 \equiv \mathcal{H}^{(F)}$.

The technical essence of the new paradigm may be found summarized in Appendix. *In nuce*, the key idea of the new formalism lies in the weakening of the standard textbook Hermiticity $H = H^\dagger$ of the Hamiltonian and in its replacement by the requirement

$$H^\dagger \, \Theta = \Theta \, H \, . \tag{12}$$

The use of this relation containing a nontrivial metric operator $\Theta \neq I$ has already been shown useful, almost twenty years ago, in nuclear physics [7]. Naturally, the same requirement applies also to the other operators $\mathcal{O}$ of observables where the rule $\mathcal{O} = \mathcal{O}^\dagger$ must again be replaced by its generalization

$$\Theta \, \mathcal{O} = \mathcal{O}^\dagger \, \Theta \, .$$

In opposite direction, the "lattice-position" matrix (9) loses its status of an observable quantity whenever the metric ceases to be a diagonal matrix. This argument (nicely explained also in Ref. [9])) further implies that also the quantity $\varrho(t, s)$ of Eq. (11) will lose its original probabilistic interpretation in general.

2.2 A return to the (smeared) positions.

A partial resolution of the puzzle of the smearing of positions was offered in Ref. [10] where the concept of the position and locality has *partially* been restored via the use of the "next to diagonal", band-matrix, $(2k+1)-$diagonal matrices of the metrics $\Theta = \Theta_k =$

$$= \left[\begin{array}{ccccccc} \theta_{11} & \theta_{12} & \ldots & \theta_{1,k+1} & 0 & \ldots & 0 \\ \theta_{21} & \theta_{22} & \theta_{23} & \ldots & \theta_{2,k+2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & 0 \\ \theta_{k+1,1} & & \ddots & \ddots & \ddots & & \theta_{N-k,N} \\ 0 & \ddots & & \ddots & \theta_{N-2,N-2} & \theta_{N-2,N-1} & \vdots \\ \vdots & \ddots & \theta_{N-1,N-k-1} & \ldots & \theta_{N-1,N-2} & \theta_{N-1,N-1} & \theta_{N-1,N} \\ 0 & \ldots & 0 & \theta_{N,N-k} & \ldots & \theta_{N,N-1} & \theta_{NN} \end{array} \right] . \tag{13}$$

The integer $k = 0, 1, \ldots, N-1$ has been interpreted there as a measure of the "smearing". In this role, its value must be kept perceivably smaller than the dimension N.

The physical meaning of the above non-diagonal metrics may be clarified using the first nontrivial $k = 1$ example in which we may rewrite metric (13) as a superposition of a positive diagonal matrix $\mathcal{D}^2$ with and upper-diagonal matrix $\alpha\, A$ and its lower-diagonal conjugate $\alpha\, A^\dagger$,

$$\Theta_1^{(N)}(a, \alpha) = \mathcal{D}^2 + \alpha\, \left(A + A^\dagger \right) . \tag{14}$$

Once we assume that the parameter α itself is sufficiently small, we may recall formula (41) of our Appendix and deduce, say, the following approximate, non-Hermitian form of the Dyson's map,

$$\Omega = \sqrt{\Theta_1^{(N)}(a, \alpha)} = \mathcal{D} + \frac{1}{2}\, \alpha\, \mathcal{D}^{-1} \left(A + A^\dagger \right) + \mathcal{O}(\alpha^2) . \tag{15}$$

In this setting, it is still most natural to follow the notation of our Appendix and to define the operator of position of our (quasi)particle by the strictly diagonal matrix (9) acting in the "inaccessible" physical Hilbert space $\mathcal{H}^{(P)}$

where the metric remains trivial, $\Theta^{(P)} = I$. Naturally, such a position operator $\hat{\mathfrak{q}}$ may be equally well represented by the nondiagonal operator

$$\hat{Q} = \Omega^{-1} \hat{\mathfrak{q}} \Omega$$

acting either in the "accessible", unitarily equivalent physical Hilbert space $\mathcal{H}^{(S)}$ (where, in our notation, $\hat{Q} = \hat{Q}^{\ddagger}$ is Hermitian) or in the unphysical and unitarily non-equivalent auxiliary Hilbert space $\mathcal{H}^{(F)}$ (where the same operator remains non-Hermitian of course, $\hat{Q} \neq \hat{Q}^{\dagger}$).

At this point our approximate formula (15) for Ω enables us to transform $\hat{\mathfrak{q}}$ into its image

$$\hat{Q} = \left[\mathcal{D} - \frac{1}{2}\,\alpha\,\mathcal{D}^{-1}\left(A + A^{\dagger}\right) + \mathcal{O}(\alpha^2)\right] \hat{\mathfrak{q}} \left[\mathcal{D} + \frac{1}{2}\,\alpha\,\mathcal{D}^{-1}\left(A + A^{\dagger}\right) + \mathcal{O}(\alpha^2)\right] =$$

$$= \mathcal{D}\hat{\mathfrak{q}}\mathcal{D} + \frac{1}{2}\,\alpha\left[\mathcal{D}\hat{\mathfrak{q}}\mathcal{D}^{-1}\left(A + A^{\dagger}\right) - \mathcal{D}^{-1}\left(A + A^{\dagger}\right)\hat{\mathfrak{q}}\mathcal{D}\right] + \mathcal{O}(\alpha^2)$$

which is, up to second-order corrections, a diagonally dominated tridiagonal matrix. The construction remains also feasible at $k > 1$, provided only that we make use of a partitioning and replace the tridiagonality of $\Theta_1^{(N)}$ by the block-tridiagonality of $\Theta_k^{(N)}$.

We may conclude that in our *ad hoc* physical Hilbert space $\mathcal{H}^{(S)}$ assigned to the cryptohermitian Hamiltonian $H^{(N)}(a)$ the original basis $|q_s\rangle$ [such that $(|q_s\rangle)_{s'} \sim \delta_{ss'}$] lost its connection with the observable position. The role of the position-eigenstates is taken over by the kets $|\chi_{q_s}\rangle$ representing our (quasi)particle(s) localized at a site s (with spatial coordinate q_s). They must be constructed as eigevectors of our cryptohermitian operator of position,

$$\hat{Q}\,|\chi_{q_s}\rangle = q_s\,|\chi_{q_s}\rangle\,, \qquad s = 1, 2, \ldots, N\,. \tag{16}$$

This is a numerical problem, possibly simplified at small α when the operator of position predominantly couples the nearest-neighbor basis states.

In the language of measurements the probability of finding the particle (or quasi-particle) in question at the $s-$th site of the lattice may still be determined by formula (11), after its appropriate amendment of course.

The form of this formula itself may even stay unchanged, provided only that we replace the old, Hermiticity-assuming definition of the wave function $\psi(t,s) := \langle q_s|\psi(t)\rangle$ (meaning the "probability-density quantity") by its new, cryptohermiticity-assuming update

$$\psi_{(\Theta_k)}(t,s) = \langle \chi_{q_s}|\Theta_k|\psi(t)\rangle = \sum_{s'=s-k}^{s+k} \langle \chi_{q_s}| \left(\Theta_k\right)_{ss'} \left(|\psi(t)\rangle\right)_{s'} . \tag{17}$$

With the time-independent parameter $a \neq a(t)$ this definition acquires, in the light of Eqs. (6) and (7), the final compact form

$$\psi_{(\Theta_k)}(t,s) = \sum_{s'=s-k}^{s+k} \langle q_s| \left(\Theta_k\right)_{ss'} \sum_{n=1}^{N} e^{iE_n t}\, L(s',a,E_n)\,. \tag{18}$$

This conclusion establishes the direct link between our cryptohermitian Hamiltonian (4) and the measurement od the position. Formula (18) indicates that the different choices of the physical metric will lead to the nonequivalent predictions of the results of the experiment. *Vice versa*, the experiments may be used, in principle, as a valid source of information fitting the parameters in phenomenological metrics [7].

3 A few mathematical prerequisites

On the level of the textbook quantum theory the main obstacle of accepting the input Hamiltonian matrix $H^{(N)} \neq \left(H^{(N)}\right)^\dagger$ (with real spectrum) lies in the technical nontriviality of making it compatible with the standard postulates of Quantum Mechanics. In Appendix we explained that this acceptance is based on the "hidden" Hermiticity

$$H^{(N)}(a) = \left(H^{(N)}(a)\right)^\ddagger \tag{19}$$

(a.k.a. cryptohermiticity [8]). This property is defined in the less common but still entirely standard Hilbert space $\mathcal{H}^{(S)}$. It may almost always be reread as the Dieudonné's [11] constraint (12) imposed in the auxiliary, friendlier Hilbert space $\mathcal{H}^{(F)}$.

For our model (4) the major technical difficulties related to the necessary construction of the right-hand-side matrices in Eq. (19) (i.e., of the eligible metrics) will be tractable due to the important observation that the metrics may *systematically* be constructed as special, $(2k+1)-$diagonal matrices (13) at *any* $k \in \{0, 1, \ldots, N-1\}$.

3.1 The exceptional local model at $k = 0$

We shall assume that the parameter a and the dimension N are both variable and, in principle, arbitrary while the number k of the nonvanishing diagonals in Eq. (13) will remain, for technical as well as interpretation reasons, fixed. Naturally, the constructive analysis of the model should start from the trivial $k = 0$. In this case it is well known that within the formalism described in Appendix, *any* tridiagonal complex matrix (8) may be made compatible with the Dieudonné's Hermitization equation $\hat{H}^\dagger\Theta = \hat{H}\,\Theta$ via the diagonal ansatz

$$\Theta_0(a) = \begin{bmatrix} \theta_{11} & & & & & \\ & \theta_{22} & & & & \\ & & \ddots & & & \\ & & & \theta_{N-1,N-1} & & \\ & & & & \theta_{NN} \end{bmatrix} \tag{20}$$

for the metric with the positive matrix elements, $\theta_{jj} > 0$, $j = 1, 2, \ldots, N$. This observation leads to the following easy consequence.

Lemma 1. *Hamiltonians $H^{(N)}(a)$ of Eq. (4) may be assigned the diagonal metrics $\Theta_0(a)$ of Eq. (20) with normalization $\theta_{11} = 1$ and with the elementary matrix elements*

$$\theta_{nn} = \frac{(n-1)!}{(a+n-1)\,(a+n-2)\ldots(a+2)\,(a+1)} \tag{21}$$

at $n = 2, 3, \ldots, N$.

Proof. In the Dieudonné's equation $\hat{H}^\dagger\Theta = \hat{H}\,\Theta$ it is easy to evaluate the

left-hand-side product of matrices for *any* tridiagonal input (8),

$$\left(\hat{H}^{(N)}\right)^{\dagger}\,\Theta=\left[\begin{array}{cccccc}
a_1^*\theta_{11} & b_2^*\theta_{22} & 0 & \ldots & 0 & 0\\
c_1^*\theta_{11} & a_2^*\theta_{22} & b_3^*\theta_{33} & 0 & \ldots & 0\\
0 & c_2^*\theta_{22} & a_3^*\theta_{33} & b_4^*\theta_{44} & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & 0\\
0 & \ldots & 0 & c_{N-2}^*\theta_{N-2,N-2} & a_{N-1}^*\theta_{N-1,N-1} & b_N^*\theta_{NN}\\
0 & \ldots & 0 & 0 & c_{N-1}^*\theta_{N-1,N-1} & a_N^*\theta_{NN}
\end{array}\right] \tag{22}$$

and to compare it with the right-hand-side tridiagonal matrix

$$\Theta\,\hat{H}^{(N)}=\left[\begin{array}{cccccc}
\theta_{11}a_1 & \theta_{11}c_1 & 0 & 0 & \ldots & 0\\
\theta_{22}b_2 & \theta_{22}a_2 & \theta_{22}c_2 & 0 & \ldots & 0\\
0 & \theta_{33}b_3 & \theta_{33}a_3 & \theta_{33}c_3 & \ddots & \vdots\\
0 & \ddots & \ddots & \ddots & \ddots & 0\\
0 & \ldots & 0 & \theta_{N-1,N-1}b_{N-1} & \theta_{N-1,N-1}a_{N-1} & \theta_{N-1,N-1}c_{N-1}\\
0 & \ldots & 0 & 0 & \theta_{NN}b_N & \theta_{NN}a_N
\end{array}\right]. \tag{23}$$

The net result of this comparison is that the diagonal elements of the Hamiltonian must be real and that the sequence of relations

$$\theta_{n+1,n+1}b_{n+1}=\theta_{nn}c_n^*\,,\qquad n=1,2,\ldots,N-1 \tag{24}$$

must be satisfied. For our present model (4) this observation immediately leads to the proof of the Lemma. □

Remark 1. The simplicity of the above result made it useful in tests of a straightforward symbolic-manipulation algorithm which we intended to use for the computer-assisted solution of the Dieudonné's equation $\hat{H}^{\dagger}\Theta=\hat{H}\,\Theta$ using the various less elementary forms of the ansatz for $(2k+1)-$diagonal metric $\Theta=\Theta_k$. Identifying the present model with the $k=0$ special case the algorithm produced the sequence of formulae

$$\theta_{22}=\frac{\theta_{11}}{a+1}\,,\qquad \theta_{33}=\frac{2\,\theta_{11}}{(a+2)\,(a+1)}\,,\quad\ldots$$

and confirmed the validity of the above closed-form rigorous result (21) and, *ipso facto*, also the reliability of the algorithm itself.

3.2 The left eigenvectors of $H^{(N)}(a)$ at $k > 1$

As long as we have to work with the phenomenological model of quantum dynamics where $\hat{H}^{(N)}(a) \neq \left(\hat{H}^{(N)}(a)\right)^\dagger$ at all N and a, it is insufficient to know just the *a priori* specified solutions (7) of the current time-independent eigenvalue problem (5). For multiple purposes it is also necessary to know the left eigenvectors $\langle\xi_n|$ of our Hamiltonian. Most often, we rather construct their duals $|\xi_n\rangle$ defined as the right, usual eigenvectors of the conjugate, unusual Hamiltonian $\left(\hat{H}^{(N)}(a)\right)^\dagger$,

$$\left(H^{(N)}(a)\right)^\dagger\,|\xi_n^{(N)}\rangle = E_n^{(N)}(a)\,|\xi_n^{(N)}\rangle\,,\qquad n=1,2,\ldots,N-1\,. \tag{25}$$

As long as we already know the energies, this is a simpler task. A rather subtle problem only emerges when we recollect that a suitable set of the left eigenbras $\langle\langle\psi_n^{(N)}|$ of matrix $H^{(N)}(a)$ has already been defined in Appendix (cf. Eq. (38)). This implies the necessity of the following proportionality relation

$$|\psi_n^{(N)}\rangle\rangle := \Theta\,|\psi_n^{(N)}\rangle = |\xi_n^{(N)}\rangle\,\kappa_n^{(N)}(a) \tag{26}$$

between the two alternative sets of the eigenvectors where the left-hand-side-state normalization is fixed while the right-hand-side-state normalization still remains open in Eq. (25).

One of the consequences of the arbitrariness of the $n-$dependent complex constants $\kappa_n^{(N)}(a)$ is particularly serious since the solutions $|\xi_n^{(N)}\rangle$ of Eq. (25) are often used in the spectral-expansion definition of the metric [12, 13],

$$\Theta = \sum_{n=0}^{N-1}\,|\psi_n^{(N)}\rangle\rangle\,\langle\langle\psi_n^{(N)}| = \sum_{n=0}^{N-1}\,|\xi_n^{(N)}\rangle\,\left|\kappa_n^{(N)}(a)\right|^2\,\langle\xi_n^{(N)}|\,. \tag{27}$$

This implies that up to an inessential overall multiplication factor (say, $\left|\kappa_0^{(N)}(a)\right|^2 > 0$), the metric (27) contains, in general, an $(N-1)-$plet of

free parameters $\left|\kappa_n^{(N)}(a)\right|^2 > 0$, $n = 1, 2, \ldots, N-1$. As a consequence, *every* given Hamiltonian $H^{(N)}(a)$ may be assigned an $(N-1)-$parametric multiplet of alternative metrics leading, generically, to independent physical predictions of the model.

For this reason it is important to have, always, a sensible *physics-based* recipe for an efficient suppression of the latter ambiguity. In field theory one often requires the observability of the so called "charge" $\mathcal{C}$ for this purpose [5]. For quantum lattices using the nearest-neighbor interaction dynamics we proposed, very recently [10], another principle of suppression of the ambiguity of the metrics which is also being used here. This recipe is based on the requirement of the minimal (or at least "tunable", controllable) non-locality of the model in question. In the latter setting, unfortunately, the *direct* use of spectral formula (27) does not work at all. At the same time, the use of the metric-multiplication definition (26) of the left eigenvectors $|\psi_n^{(N)}\rangle\rangle$ may be recommended as facilitated by the $(2k+1)-$diagonal band-matrix structure of the metric, especially at the not too large k.

4 The method of construction of the multi-diagonal metrics

4.1 The sparse-matrix-expansion ansatz.

Our preliminary computer-assisted experiments with solving Dieudonné's Eq. (12) with Hamiltonian (4) revealed that its $(2k+1)-$diagonal matrix solutions (13) might be sought in a specific linear-superposition form

$$\Theta = \Theta_k^{(N)}(a) = \Theta_{(k,\alpha 1,\alpha_2,\ldots,\alpha_k)}^{(N)}(a) = \Theta_0^{(N)}(a) + \sum_{j=1}^{k} \alpha_j \, \mathcal{P}_j^{(N)}(a) \qquad (28)$$

with $k \in \{1, 2, \ldots, N-1\}$ and with the diagonal metric of paragraph 3.1 accompanied by a $k-$plet of $(2j+1)-$diagonal real and symmetric matrices $\mathcal{P}_j^{(N)}(a)$. In fact, just the diagonal matrix $\Theta_0^{(N)}(a)$ is a true, positive-definite

metric. Every other matrix $\mathcal{P}_j^{(N)}(a)$ is only expected *individually* compatible with the Dieudonné's constraint,

$$\left(H^{(N)}(a)\right)^\dagger\,\mathcal{P}_j^{(N)}(a)=\mathcal{P}_j^{(N)}(a)\,H^{(N)}(a)\,,\qquad j=1,2,\ldots,k\,. \tag{29}$$

In this overall framework the core of the exhaustive construction of the expansion (28) at a given non-locality k lies, obviously, *in* the constructions of all of the pseudometrics (29) with $j\leq k$ *and in* having the real expansion parameters α_j sufficiently small to keep the necessary positive definiteness of the diagonally-dominated sum (28) unbroken.

4.2 The extrapolation method of solving Eq. (29).

In what follows the systematic and explicit construction of pseudometrics $\mathcal{P}_j^{(N)}(a)$ will be performed for all $j\leq 3$. This construction will proceed in three steps.

In a preparatory step we select an integer value of subscript $j\geq 1$, insert a general $(2j+1)-$diagonal-matrix ansatz for $\mathcal{P}_j^{(N)}(a)$ in Eq. (29) and, using a computer-assisted trial-and-error strategy, fine-tune an ansatz for $\mathcal{P}_j^{(N)}(a)$ in such a way that a maximum of its matrix elements not lying on its outer diagonals is being set equal to zero.

In the second step of the algorithm we select a few $N\leq N_0$ and, via the computer-assisted closed-form solution of Eq. (29) we construct *all* of the corresponding matrix elements $\theta_{mm'}=\theta_{mm'}^{(N,j)}(a)$ of $\mathcal{P}_j^{(N)}(a)$. In our concrete model this computer-assisted "experiment" revealed, after certain symbolic-manipulation factorizations and simplifications of the originally quite tedious results, the prevailing $N-$independence and sufficiently transparent and nicely factorizable $a-$dependence of the individual matrix elements of our separate pseudometric matrices $\mathcal{P}_j^{(N)}(a)$, at the first few smallest j at least.

In the third step the resulting sample of elements must be perceived and reanalyzed as a set which is generated by the comparatively elementary linear

algebraic set of equations (29). These equations always possess, at any fixed N and j, an implicitly recurrent structure. Such a reinterpretation of the algebra opens the possibility of skipping the intermediate step of the *explicit* determination of the individual recurrences as sampled, at $j = 0$, by Eq. (24) above. Especially at the higher values of j such an intermediate step did prove prohibitively complicated and, at the same time, redundant, especially due to the elementary nature of our Hamiltonian in question.

Thus, in the third step we replace the tedious multidimensional recurrent generation of the closed algebraic formulae for the unknown multiplets of elements $\theta_{mm'}$ by the extrapolation technique. In our model the latter recipe proved more efficient than the direct use of recurrences even at the lowest values of j. Naturally, having the closed form of extrapolation at our disposal at last, the final rigorous proof of its general compatibility with Eq. (29) by direct insertion remains straightforward.

5 The model with minimal smearing ($k = 1$).

In the general family of $\alpha-$dependent tridiagonal metrics

$$\Theta_1^{(N)}(a,\alpha) = \Theta_0^{(N)}(a) + \alpha\,\mathcal{P}_1^{(N)}(a) \tag{30}$$

"numbered" by a real and not too large variable α we may assume, without any loss of generality, that the matrix $\mathcal{P}_1^{(N)}(a)$ is real and symmetric and that without loss of generality its element θ_{11} may be chosen as vanishing,

$$\mathcal{P}_1^{(N)}(a) = \left[\begin{array}{ccccccccc}
0 & \theta_{12} & & & & & & & \\
\theta_{12} & \theta_{22} & \theta_{23} & & & & & & \\
 & \theta_{23} & \theta_{33} & \theta_{34} & & & & & \\
 & & \ddots & \ddots & & \ddots & & & \\
 & & & \theta_{N-3,N-2} & \theta_{N-2,N-2} & \theta_{N-2,N-1} & & & \\
 & & & & \theta_{N-2,N-1} & \theta_{N-1,N-1} & \theta_{N-1,N} & & \\
 & & & & & \theta_{N-1,N} & \theta_{NN} & &
\end{array}\right]. \tag{31}$$

One of the most unexpected observations made during the brute-force construction of this pseudometric appeared to be the cutoff-independence of the result and, in particular, of the "last" element $\theta_{NN} \neq 0$. Thus the "initial" list of the elements

$$\theta_{22} = -\frac{2}{a+1}\,, \quad \theta_{33} = -\frac{8}{(a+2)\,(a+1)}\,,$$

$$\theta_{44} = -\frac{36}{(a+3)\,(a+2)\,(a+1)}\,, \quad \theta_{12} = 1\,,$$

$$\theta_{23} = \frac{2}{a+1}\,, \quad \theta_{34} = \frac{6}{(a+2)\,(a+1)}$$

obtained at $N \leq N_0 = 4$ proved sufficient for the extrapolation and for the final formulation of the general result.

Lemma 2. *Hamiltonians $H^{(N)}(a)$ of Eq. (4) may be assigned the tridiagonal family of metrics (30). The elementary matrix elements of pseudometric (31) are given by the cutoff-insensitive formulae*

$$\theta_{nn} = \frac{(n-1)!}{(a+n-1)\,(a+n-2)\ldots(a+2)\,(a+1)}$$

at $n = 2, 3, \ldots, N$, and

$$\theta_{nn+1} = \frac{n!}{(a+n-1)\,(a+n-2)\ldots(a+2)\,(a+1)}$$

at $n = 1, 2, \ldots, N-1$.

Proof. In a parallel to the constructive proof of Lemma 1 the tridiagonal-metric-generated recurrent relations remain sufficiently transparent to admit the explicit identification and the straightforward proof of validity of their closed solutions by mathematical induction not only for our model (4) but for any tridiagonal input Hamiltonian (8). The details are left to the readers. □

5.1 The verification of extrapolation hypotheses

The computation technique which we use here may be characterized as an interactive algorithm based on a systematic computer-assisted verifications

of the series of amended extrapolation hypotheses. It will remain applicable, *mutatis mutandis*, at any $k > 1$, throughout our forthcoming constructions.

The key technical problem will always lie in the determination of an appropriate ansatz for pseudometrics. In practice, the only way of finding the optimal ansatz seems to lie in a patient, brute-force solution of Eq. (12) at a sequence of the smallest dimensions N. That's what we will always have to do. Without the help of MAPLE, such a task would be rather difficult.

In an illustrative verification of the result presented in Lemma 2 we may compare its predictions, say, with their two computer-generated counterparts

$$\theta_{45} = \frac{24}{(a+3)\,(a+2)\,(a+1)}\,, \quad \theta_{55} = -\frac{192}{(a+4)\,(a+3)\,(a+2)\,(a+1)}\,.$$

This comparison reconfirms the validity of Lemma 2. Moreover, these and similar "redundant" formulae may offer an insight and background for the estimates, say, of the numerical magnitude of the eigenvalues of the metric at larger parameters α and/or a. In the context of a different quantum-lattice model a confirmation of feasibility of such a nonperturbative search for the strong-coupling boundaries of the domain of positivity of the metric was mediated, e.g., by Table I of Ref. [14].

An additional merit of the interactive amendment of the tentatively extrapolated formulae via the *a posteriori* computer-assisted verification has been found in its speed. The initial tedious algorithm of the lengthy direct construction gets easily amended in the light of the extrapolation so that the calculations at the higher dimensions prove, paradoxically, quicker. Giving, in our illustrative example, the sequence of further elements

$$\theta_{56} = \frac{120}{(a+4)\,(a+3)\,(a+2)\,(a+1)}\,,$$

$$\theta_{66} = -\frac{1200}{(a+5)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}$$

(etc) which further confirm the reliability of the formulae and open the way to the perceivably simplified proofs using direct insertions.

6 Pentadiagonal metrics ($k = 2$)

In the general pentadiagonal metric of our model (4),

$$\Theta_2^{(N)}(a,\alpha,\beta) = \Theta_0^{(N)}(a) + \alpha\,\mathcal{P}_1^{(N)}(a) + \beta\,\mathcal{P}_2^{(N)}(a) \tag{32}$$

the only unknown matrix $\mathcal{P}_2^{(N)}(a)$ must again be real and symmetric, normalized, say, by the choice of $\theta(1,3) = 1$ and finally, without loss of generality, admitting that the elements θ_{11} and θ_{12} vanish,

$$\mathcal{P}_2^{(N)}(a) = \begin{bmatrix}
0 & 0 & \theta_{13} & & & & & \\
0 & \theta_{22} & \theta_{23} & \theta_{24} & & & & \\
\theta_{13} & \theta_{23} & \theta_{33} & \theta_{34} & \theta_{35} & & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & \theta_{N-4,N-2} & \theta_{N-3,N-2} & \theta_{N-2,N-2} & \theta_{N-2,N-1} & \theta_{N-2,N} \\
 & & & \theta_{N-3,N-1} & \theta_{N-2,N-1} & \theta_{N-1,N-1} & \theta_{N-1,N} \\
 & & & & \theta_{N-2,N} & \theta_{N-1,N} & \hat{t}_{NN}^{(N)}
\end{bmatrix}. \tag{33}$$

The hat-superscripted matrix element $\hat{t}_{NN}^{(N)}$ is exceptional and must be considered manifestly cutoff-dependent. This has been revealed during the first-step calculations which gave the initial list of the cutoff-insensitive elements

$$\theta_{22} = \frac{a+2}{a+1}\,, \quad \theta_{33} = \frac{4\,(a+5)}{(a+2)\,(a+1)}\,, \quad \theta_{44} = \frac{18\,(a+8)}{(a+3)\,(a+2)\,(a+1)}\,,$$

$$\theta_{55} = \frac{96\,(a+11)}{(a+4)\,(a+3)\,(a+2)\,(a+1)} \neq \hat{t}_{55}^{(5)} = \frac{36\,(a+21)}{(a+4)\,(a+3)\,(a+2)\,(a+1)}\,,$$

$$\theta_{23} = -\frac{4}{a+1}\,, \quad \theta_{34} = -\frac{24}{(a+2)\,(a+1)}\,, \quad \theta_{45} = -\frac{144}{(a+3)\,(a+2)\,(a+1)}\,,$$

$$\theta_{13} = 1\,, \quad \theta_{24} = \frac{3}{a+1}\,, \quad \theta_{35} = \frac{12}{(a+2)\,(a+1)}\,,$$

leading, by extrapolation and by its subsequent tests at a few $N > N_0 = 5$, to the following general result.

Lemma 3. *Hamiltonians $H^{(N)}(a)$ of Eq. (4) may be assigned the pentadiagonal family of metrics (32). The cutoff-insensitive matrix elements of*

pseudometric (33) are given by formula

$$\theta_{nn} = \frac{(n-1)\,(n-1)!\,(a+3\,n-4)}{(a+n-1)\,(a+n-2)\ldots(a+2)\,(a+1)}$$

at $n = 2, 3, \ldots, N$, *and by*

$$\theta_{nn+1} = -\frac{2\,(n-1)\,n!}{(a+n-1)\,(a+n-2)\ldots(a+2)\,(a+1)}$$

at $n = 2, 3, \ldots, N-1$, *and by*

$$\theta_{nn+2} = \frac{(n+1)!}{2\;(a+n-1)\,(a+n-2)\ldots(a+2)\,(a+1)}$$

at $n = 2, 3, \ldots, N-2$.

Proof. We leave the proof by direct insertion to the readers. □

Remark 2. The determination of the missing exceptional-element sequence $\hat{t}_{NN}^{(N)}$ requires the use of a different approach, outlined in section 8 below.

7 The metrics with seven diagonals, $k = 3$.

In the seven-diagonal metrics

$$\Theta_3^{(N)}(a,\alpha,\beta,\gamma) = \Theta_0^{(N)}(a) + \alpha\,\mathcal{P}_1^{(N)}(a) + \beta\,\mathcal{P}_2^{(N)}(a) + \gamma\,\mathcal{P}_3^{(N)}(a) \tag{34}$$

the missing real and symmetric component $\mathcal{P}_3^{(N)}(a)$ may be constructed along the same lines as above, starting from the assumptions $\theta_{11} = \theta_{22} = 0$, $\theta(1,2) = \theta(1,3) = 0$ and $\theta(1,4) = 1$ and from the heptadiagonal ansatz

$$\mathcal{P}_3^{(N)}(a) = \begin{bmatrix} 0 & 0 & 0 & \theta_{14} & & & \\ 0 & 0 & \theta_{23} & \theta_{24} & \ddots & & \\ 0 & \theta_{23} & \theta_{33} & \theta_{34} & \ddots & \theta_{N-4,N-1} & \\ \theta_{14} & \theta_{24} & \theta_{34} & \theta_{44} & \ddots & \theta_{N-3,N-1} & \theta_{N-3,N} \\ & \theta_{25} & \ddots & \ddots & \ddots & \theta_{N-2,N-1} & \theta_{N-2,N} \\ & & \ddots & \theta_{N-3,N-1} & \theta_{N-2,N-1} & \theta_{N-1,N-1} & \hat{t}_{N-1,N} \\ & & & \theta_{N-3,N} & \theta_{N-2,N} & \hat{t}_{N-1,N} & \hat{t}_{NN}^{(N)} \end{bmatrix}. \tag{35}$$

We now encounter the two specific, hat-superscripted matrix elements $\hat{t}^{(N)}_{NN}$ and $\hat{t}^{(N)}_{N-1,N}$ which must be constructed in different manner (cf. section 8 below).

For the purposes of extrapolation we evaluated, this time, the following $N_0 = 6$ set of the cutoff-insensitive matrix elements,

$$\theta_{33} = -8\,\frac{a+3}{(a+2)\,(a+1)}\,,\quad \theta_{44} = -24\,\frac{3\,a+14}{(a+3)\,(a+2)\,(a+1)}\,,$$

$$\theta_{55} = -192\,\frac{3\,a+19}{(a+4)\,(a+3)\,(a+2)\,(a+1)}\,,$$

$$\theta_{66} = -4800\,\frac{a+8}{(a+5)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}\,,$$

$$\theta_{23} = \frac{a+3}{a+1}\,,\quad \theta_{34} = 6\,\frac{a+8}{(a+2)\,(a+1)}\,,$$

$$\theta_{45} = 36\,\frac{a+13}{(a+3)\,(a+2)\,(a+1)}\,,$$

$$\theta_{56} = 240\,\frac{a+18}{(a+4)\,(a+3)\,(a+2)\,(a+1)}\,,$$

$$\theta_{24} = -\frac{6}{a+1}\,,\quad \theta_{35} = -\frac{48}{(a+2)\,(a+1)}\,,\quad \theta_{46} = -\frac{360}{(a+3)\,(a+2)\,(a+1)}\,,$$

$$\theta_{14} = 1\,,\theta_{25} = \frac{4}{a+1}\,,\quad \theta_{36} = \frac{20}{(a+2)\,(a+1)}$$

leading, by extrapolation and its subsequent verification, to the following general result.

Lemma 4. *Hamiltonians $H^{(N)}(a)$ of Eq. (4) may be assigned the hepta-diagonal family of metrics (34). The cutoff-insensitive matrix elements of pseudometric (35) are given by formula*

$$\theta_{nn} = -\frac{2\,(n-1)\,(n-2)\,(n-1)!\;(a+5\,n-6)}{3\;(a+n-1)\,(a+n-2)\ldots(a+2)\,(a+1)}$$

at $n = 2, 3, \ldots, N$, and by

$$\theta_{nn+1} = \frac{(n-1)\,n!\,(a+5\,n-7)}{2\;(a+n-1)\,(a+n-2)\ldots(a+2)\,(a+1)}$$

at $n = 2, 3, \ldots, N-1$, and by

$$\theta_{nn+2} = -\frac{(n-1)\,(n+1)!}{(a+n-1)\,(a+n-2)\ldots(a+2)\,(a+1)}$$

at $n = 2, 3, \ldots, N-2$, *and by*

$$\theta_{nn+3} = \frac{(n+2)!}{6\ (a+n-1)\,(a+n-2)\ldots(a+2)\,(a+1)}$$

at $n = 2, 3, \ldots, N-3$.

Proof. Again, we leave the proof by direct insertion to the readers. □

8 The exceptional, cut-off-dependent matrix elements $\hat{t}^{(N)}$

8.1 The construction of $\hat{t}^{(N)}_{NN}$ at $k = 2$.

The mathematical nature of the determination of the exceptional elements in ansatz (33) is special. Firstly, it is fairly clumsy to think about their values as resulting from recurrences since they vary with N themselves. Secondly, the brute-force collection of data needed for extrapolation with respect to N is much more time-consuming. At the same time, one can extrapolate these data with respect to the integer variable N almost as easily as in the case of the other matrix elements.

In such a situation we decided not to search for the rigorous proofs. Our readers will only be offered here the empirically multiply reconfirmed (i.e., for us, credible enough) results of the computer-assisted (though still rather time-consuming) extrapolations.

Conjecture 1. *At* $j = 2$ *and at all* $N = 3, 4, \ldots$ *formula*

$$\tilde{t}^{(N)}_{NN} = \frac{(N-2)\,(N-1)!\ (a+5\,N-4)}{2\ (a+N-1)\,(a+N-2)\ldots(a+2)\,(a+1)}$$

defines the "last missing matrix element" $\hat{t}^{(N)}_{NN}$ *of pseudometric (33).*

Remark 3. The validity of the latter formula (obtained by the extrapolation from the data at $N \leq N_0 = 5$) has been made plausible, for us, by the direct MAPLE-mediated evaluation of the subsequent values up to

$$\tilde{t}^{(9)}_{99} = \frac{141120\ (a+41)}{(a+8)\,(a+7)\,(a+6)\,(a+5)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}\,.$$

8.2 The construction of $\hat{t}^{(N)}_{NN}$ at $k = 3$.

The computer-assisted solution of Eq. (29) enabled us to evaluate the following set of results of symbolic manipulations,

$$\hat{t}^{(4)}_{44} = -16\,\frac{a+7}{(a+3)\,(a+2)\,(a+1)}$$

$$\hat{t}^{(5)}_{55} = -16\,\frac{11\,a+103}{(a+4)\,(a+3)\,(a+2)\,(a+1)}$$

$$\hat{t}^{(6)}_{66} = -240\,\frac{7\,a+82}{(a+5)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}$$

$$\hat{t}^{(7)}_{77} = -960\,\frac{17\,a+239}{(a+6)\,(a+5)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}$$

which proved extrapolated as follows.

Conjecture 2. *At $j = 3$ and at all $N = 3, 4, \ldots$ formula*

$$\hat{t}^{(N)}_{NN} = -\frac{(N-3)\,(N-1)!\,[(3\,N-4)\,a+7\,N^2-16\,N+8]}{3\,(a+N-1)\,(a+N-2)\ldots(a+2)\,(a+1)}$$

defines the diagonal missing matrix element of pseudometric (35).

Remark 4. Again, for us, the plausibility of this conjecture has been enhanced by the MAPLE-mediated evaluation of the subsequent values up to

$$\hat{t}^{(9)}_{99} = -80640\,\frac{23\,a+431}{(a+8)\,(a+7)\,(a+6)\,(a+5)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}\,.$$

A deeper meaning of the display of similar formulae may be seen not only in the misprint-control in the general formulae but also in the facilitated future identifications of the underlying recurrences if any. In addition, the transparency of the formulae (and, in particular, of their $a-$dependence) also underlines the formal simplicity of the metric itself, especially when we compare its matrix elements, say, with their complicated polynomial analogues as obtained in Ref. [10].

8.3 The construction of $\hat{t}^{(N)}_{N-1N}$ at $k = 3$.

In a way paralleling the preceding section we started from the data

$$\hat{t}^{(4)}_{34} = 2\,\frac{a+16}{(a+2)\,(a+1)}$$

$$\hat{t}^{(5)}_{45} = 16\,\frac{a+23}{(a+3)\,(a+2)\,(a+1)}$$

$$\hat{t}^{(6)}_{56} = 120\,\frac{a+30}{(a+4)\,(a+3)\,(a+2)\,(a+1)}$$

$$\hat{t}^{(7)}_{67} = 960\,\frac{a+37}{(a+5)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}$$

and formulated our last extrapolation hypothesis.

Conjecture 3. *At $j = 3$ and at all $N = 3, 4, \ldots$ formula*

$$\hat{t}^{(N)}_{N-1N} = \frac{(N-3)\,(N-1)!\,(a+7\,N-12)}{3\,(a+N-2)\,(a+N-3)\ldots(a+2)\,(a+1)}$$

defines the last missing off-diagonal matrix element of pseudometric (35).

Remark 5. We may again display the two subsequent MAPLE-generated quantities, viz.,

$$\hat{t}^{(8)}_{78} = 8400\,\frac{a+44}{(a+6)\,(a+5)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}$$

and

$$\hat{t}^{(9)}_{89} = 80640\,\frac{a+51}{(a+7)\,(a+6)\,(a+5)\,(a+4)\,(a+3)\,(a+2)\,(a+1)}\,.$$

The inspection of these formulae demonstrates that our model with the Laguerre-polynomial wave functions remains extrapolation-friendly even at $k = 3$. This is in contrast with the observations made in Ref. [14] where the complexity of the elements of metrics grew only too quickly between $k = 2$ and $k = 3$. In *loc. cit.*, we even failed to find a reasonable $k = 3$ formulae, due to a loss of any obvious guidance for the extrapolations. In the present model such an overall pattern seems to stay unchanged. Hence, one could also expect the reasonable feasibility of further, $k \geq 4$ constructions.

9 Summary and discussion

A family of exactly solvable $N-$site quantum lattices with a non-Hermitian nearest-neighbor interaction was proposed and studied. The energies appeared real so that each Hamiltonian has been made self-adjoint, i.e., standard and physical in an *ad hoc* Hilbert space $\mathcal{H}^{(S)}$ where the inner product was defined via a metric Θ. The complete set of the eligible metrics has been shown numbered by a multiindex $(k, \alpha_1, \ldots, \alpha_k)$ in which the "degree of non-locality" $k \in \{0, 1, \ldots, N-1\}$ indicates that every $\Theta = \Theta_{(k,\vec{\alpha})}$ is a $(2k+1)-$diagonal matrix. The other free parameters forming the $k-$plets $\vec{\alpha}$ must only be real and sufficiently small (otherwise, the metric could cease to be positive definite). In closed form the metrics were constructed and displayed for $k = 0, k = 1, k = 2$ and $k = 3$.

One of the main merits of the present quantum lattice model lies in a maximal suppression of its numerical aspects. First of all, the model does not need numerical methods for the solution of the time-independent Schrödinger equation. The reason is that the ket-eigenvectors of our special, solvable Hamiltonian $H^{(N)}(a)$ were simply selected in advance (cf. Eqs. (1) or (7)). We believe that from the point of view of flexibility of the model such an apparently very strong *a priori* constraint has been more than compensated by the multiplicity of the eligible metrics $\Theta^{(N)}_{k,\vec{\alpha}}(a) \neq I$.

In a phenomenological context we explained that our model offers a new pattern of a "smearing" of the position of the lattice sites in the manner explained in paragraph 2.2 above. Such a form of nonlocality has been shown mediated by the metric. In accord with the "weakly nonlocal" interpretation of quantum systems as advocated in our recent paper [10] we showed how the "site" of the lattice gets smeared over $2k+1$ neighbors, provided only that the range of smearing k is not too large.

In practice people rarely select *both* the suitable Hamiltonian $H = H^{(input)}$ *and* the metric $\Theta = \Theta^{(input)}$ as the two independent sources of informa-

tion about quantum dynamics. An important amendment of this limitation emerged in the literature cca twelve years ago when a real boom of interest in quantum models with nontrivial metrics $\Theta \neq I$ has been initiated by Bender and his colleagues [5, 15, 16, 17]. These authors emphasized that the use of $\Theta \neq I$ seems strongly motivated in quantum field theory.

Incidentally, in a way explained by Mostafazadeh [9] and emphasized by Jones [18] the latter studies were solely using, in our present language, the long-range metrics $\Theta_k^{(N)}$ with the maximal possible subscript $k = N-1$. This made the $k = N - 1$ recipe inapplicable in the unitary theory of scattering [19, 20]. The core of the problem has been identified with the locality of the interaction [21]. The restoration of the manifest unitarity of quantum scattering has only been achieved via the use of nonlocal interactions [22].

Fortunately, the difficulties of this type were successfully dealt with in various chain-interaction phenomenological models (cf. [23]). A number of constructive $\Theta \neq I$ results appeared in the context of Bose-Hubbard [24] or Friedrichs-Fano-Anderso [25]) models. The appeal of these models in condensed-matter physics resulted in detailed descriptions of the tightly bound lattices of electrons [26]), of the XXZ spin chains [27]) and, last but not least, of certain sophisticated experiments in optics [28, 29].

In this context the distinctive feature of our present model lies in its exact solvability. Another relevant property of our present model is that its spectrum is safely real at all $a > 0$ (cf. Eq. (6)). We believe that such a spectrum might find useful applications, say, in a purely phenomenologically motivated fitting of measured energies.

Although the similar idea has been also proposed and illustrated, via another model, in Ref. [14], the present spectrum (sampled, for reference purposes, in Table 1 above) looks much better suited for the fitting purposes. For various values of parameter a the present energy levels are regularly spread over a subinterval of the real axis which grows with the growth of the matrix dimension N. In contrast, the distribution of the bound-state

Table 2: A sample of the dimension- and parameter-dependence of the energy spectra $\{E_n(a)\}$ for the quantum-lattice Hamiltonian of Ref. [14].

parameters		energies				
N	a	$E_0(a)$	$E_1(a)$	...	$E_{N-2}(a)$	$E_{N-1}(a)$
6	1.0	-.6441855418	-.3709690601	...	.3709690601	.6441855418
	2.0	-.5622585222	-.3463446402	...	.3463446402	.5622585222
	3.0	-.5083600312	-.3244642920	...	.3244642920	.5083600312
9	1.0	-.6443127436	-.3985980302	...	.3985980302	.6443127436
	2.0	-.5626496595	-.3794465124	...	.3794465124	.5626496595
	3.0	-.5091239690	-.3616352067	...	.3616352067	.5091239690

energies of the model of Ref. [14] (sampled here by Table 2 for comparison) seems handicapped not only by its restriction to a fixed interval $(-1, 1)$ but, more seriously, by its not-well-motivated symmetry with respect to its center (shifted, conveniently, to the origin in Ref. [14]) and, even more seriously, by the weaker sensitivity of its extreme values to the changes of the dimension parameter N.

Besides the above-emphasized potential *physical* relevance of the exact solvability of our present family of Laguerrian bound-state models we would like to mention, in the conclusion, also the purely mathematical appeal and consequences of our closed-form constructions.

Firstly, the unusual though still fully non-numerical solvability of our model could serve, in principle at least, in the role of the initial zeroth approximation in perturbation theory. After all, just a very few exactly solvable models with metrics $\Theta \neq I$ exist on the market. More attention has only been paid to a few exceptionally simple quantum lattices with certain extremely elementary point-like interactions [30, 31].

Secondly, one should emphasize that it was certainly unexpected that the innocent-looking freedom in the choice of the proportionality constants

$\kappa_n^{(N)}(a)$ in relation (26) proved able to render the general definition of metric (27) compatible with the fairly strong *simplification* requirement of the sparse-matrix structure of its $(2k+1)-$diagonal representations $\Theta_{k,\vec{\alpha}}^{(N)}(a)$ as specified by Eqs. (13) and (28).

From the point of view of the theory of classical orthogonal polynomials the latter result implies that every set of column vectors (7) of Laguerre polynomials may be assigned *many different* biorthogonalized sets of row vectors. They may be formed of the "ketket" eigenvectors $|\psi\rangle\rangle$ of $\left[H^{(N)}(a)\right]^\dagger$ and classified by the same multiindex $\{k,\vec{\alpha}\}$ as the metrics, therefore. Indeed, we have

$$|\psi_n^{(N)}(a)\rangle\rangle = \Theta_{k,\vec{\alpha}}^{(N)}(a)\,|\psi_n^{(N)}(a)\rangle$$

or, in the componentwise notation of section 2.2,

$$\left(|\psi_n^{(N)}(a)\rangle\rangle\right)_s = \sum_{s'=\max(1,s-k)}^{\min(N,s+k)} \left(\Theta_{k,\vec{\alpha}}^{(N)}(a)\right)_{ss'} \left(|\psi_n^{(N)}(a)\rangle\right)_{s'} \tag{36}$$

where $s=1,2,\ldots,N$ and where matrices $\Theta_{k,\vec{\alpha}}^{(N)}(a)$ are now at our disposal in closed form (28), for $0\leq k\leq 3$ at least.

Thirdly, the variability of the $k-$parametric $(2k+1)-$diagonal metrics $\Theta_{k,\vec{\alpha}}^{(N)}(a)$ could prove useful during the model-building in which the energies are prescribed by our "Laguerrean" input Hamiltonian $H^{(N)}(a)$ but in which *several other* matrices of observables might be required selfadjoint in at least one of the eligible Hilbert spaces of states $\mathcal{H}_{k,\vec{\alpha}}^{(S)}$, typically, via the fitting of the k free parameters $\vec{\alpha}$.

Fourthly, on the level of methods the present approach to solvability could open new ways of circumventing the difficulties which emerged during the study of scattering with $H\neq H^\dagger$ [18, 19]. The use of the non-numerical, exactly solvable models with tridiagonal Hamiltonians and large $N\to\infty$ seems to have been quite effective in this context [23, 32].

Fifthly, the quick progress achieved during the study of $\mathcal{PT}-$symmetric solvable differential equations [33] has not been followed by the sufficiently

rapid progress in understanding of the related metrics. One of the reasons lies in the virtually prohibitive technical obstacles [9]. In contrast, the recent transition to $\mathcal{PT}-$symmetric difference Schrödinger equations has been accompanied by the comparatively quick success in finding the comparatively extensive sets of metrics Θ compatible with a given H [10, 32, 34].

Sixthly, a deeper understanding of the problem of ambiguity of the assignment of a metric to an input Hamiltonian has been reached here via the matrix Schrödinger equations. We pointed out that the old paradoxes [7] are finding new resolutions, say, due to the feasibility of a "smearing" of the coordinates at $N < \infty$. Due to the variability of k in our model we showed how the smeared localization could be studied by the experimental measurements, in principle at least (cf. also the possible application of this idea to cryptohermitian guantum graphs as mentioned in Refs. [34, 35]).

Last but not least, our continuing attention paid to the exactly solvable models could clarify the possibilities of the practical use of the concept of the hidden Hermiticity (cryptohermiticity) also in connection with the unitary time evolution where the time-dependent metrics emerge. As long as this idea leads to many new technical challenges [36], our present non-numerical model could re-demonstrate, in the nearest future, its relevance and importance also in this new context.

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Appendix: Three Hilbert spaces $\mathcal{H}^{(F)}$, $\mathcal{H}^{(S)}$ and $\mathcal{H}^{(P)}$ and the concept of hidden Hermiticity

Whenever a non-Hermitian Hamiltonian matrix $H \neq H^\dagger$ with real spectrum is declared physical, the Hilbert space of states *cannot* be represented by the most common $N-$dimensional version of ℓ^2. The latter space may only play an auxiliary role (this space without any immediate physical interpretation will be denoted here by the symbol $\mathcal{H}^{(F)}$ where the superscript $^{(F)}$ may stand for "friendly" as well as for "false" [8]).

For the work with Hamiltonians $H \neq H^\dagger$ there exists a trick introduced in physics, presumably, by Scholtz et al [7]. Its essence lies in the replacement of inappropriate $\mathcal{H}^{(F)}$ by a unitarily non-equivalent, "standardized" Hilbert space $\mathcal{H}^{(S)}$. By construction, the two Hilbert spaces share the same set of kets (in our notation, $|\psi\rangle \in \mathcal{V}^{(S)} = \mathcal{V}^{(F)} = \mathcal{V}$). The only distinguishing feature is that the usual Hermitian conjugation, i.e., the most conventional transition to the duals, $\mathcal{T}^{(F)}: \mathcal{V} \to \mathcal{V}'$ (mapping

$$\mathcal{T}^{(F)}: \ |\psi\rangle \ \to \ \langle\psi| \qquad \text{in} \qquad \mathcal{H}^{(F)} \tag{37}$$

in the current Dirac's notation) must be replaced, in $\mathcal{H}^{(S)}$, by the generalized conjugation $\mathcal{T}^{(S)}: \mathcal{V}\,(=\mathcal{V}^{(S)}) \to [\mathcal{V}']^{(S)}$. The latter mapping generates the different duals defined in terms of a suitable matrix Θ of metric [8],

$$\mathcal{T}^{(S)}: \ |\psi\rangle \ \to \ \langle\psi^{(S)}| := \langle\!\langle\psi| := \langle\psi|\,\Theta \qquad \text{in} \qquad \mathcal{H}^{(S)}\,. \tag{38}$$

An instructive illustration of this scenario may be found in paper [7] where the kets $|\psi\rangle \in \mathcal{V} = \mathcal{V}^{(F)} = \mathcal{V}^{(S)}$ were just bosonic, imperfect representatives of the true nuclear (i.e., *fermionic*) state vectors

$$|\psi\succ = \Omega\,|\psi\rangle \ \in \ \mathcal{W} \tag{39}$$

which lied in the textbook Hilbert space $\mathcal{H}^{(P)}$ with metric $\Theta^{(P)} = I$. The so called Dyson's map $\Omega: \mathcal{V} \to \mathcal{W}$ of bosons upon fermions (39) was chosen non-unitary so that $\prec\psi_a|\psi_b\succ \neq \langle\psi_a|\psi_b\rangle$ since $\Omega^\dagger \neq \Omega^{-1}$.

In opposite direction the non-unitarity of the map Ω enables us to require that

$$\prec \psi_a|\psi_b\succ = \langle\psi_a|\Omega^\dagger\Omega|\psi_b\rangle = \langle\langle\psi_a|\psi_b\rangle\,. \tag{40}$$

This formula connects the Dyson's map and the metric by the most important relation

$$\Theta = \Omega^\dagger\Omega\,. \tag{41}$$

The same picture of reality is obtained in *both* of the alternative physical Hilbert spaces of states $\mathcal{H}^{(P)}$ and $\mathcal{H}^{(S)}$ which are, by construction, unitarily equivalent.

In the light of definition (39) the result of action of H upon $|a\rangle$, i.e., a new vector $|b\rangle = H|a\rangle$ appears in equivalent relation $\Omega^{-1}|b\succ = H\Omega^{-1}|a\succ$, i.e., we have $|b\succ = \mathfrak{h}|a\succ$ where we abbreviated $\mathfrak{h} = \Omega H\Omega^{-1}$. The latter image of Hamiltonian must be self-adjoint in $\mathcal{H}^{(P)}$, i.e., the operator $\mathfrak{h} = \Omega H\Omega^{-1}$ must be equal to its conjugate in $\mathcal{H}^{(P)}$, viz., to $\mathfrak{h}^\dagger = (\Omega^{-1})^\dagger\, H^\dagger\Omega^\dagger$. Such a constraint imposed in $\mathcal{H}^{(P)}$ is strictly equivalent to formula (12) valid in the other two spaces $\mathcal{H}^{(S)}$ and $\mathcal{H}^{(F)}$.

For the Hamiltonian operator we have to distinguish between its apparent non-Hermiticity $H \neq H^\dagger$ in unphysical space $\mathcal{H}^{(F)}$ and the true, "hidden" Hermiticity in the physical Hilbert space $\mathcal{H}^{(S)}$ a.k.a. cryptohermiticity. In the latter space the Hermitian conjugate of H (let us denote it by the symbol $H^\ddagger$) is defined, consistently, by the prescription

$$H^\ddagger := \Theta^{-1}H^\dagger\,\Theta \quad \text{in} \quad \mathcal{H}^{(S)}\,. \tag{42}$$

In this language, Eq. (12) is precisely the disguised condition $H = H^\ddagger$ of the Hermiticity of the Hamiltonian at a fixed Θ. This scenario has been used, e.g., in Ref. [7]. On the contrary, whenever we select the concrete form of matrix H in advance, relations (12) must be reread as the Dieudonné's (incomplete) set of linear equations for the matrix elements of $\Theta = \Theta(H)$. This is precisely the approach used in our present paper.